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Universality of 2d causal dynamical triangulations

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Abstract

The formalism of Causal Dynamical Triangulations (CDT) attempts to provide a non-perturbative regularization of quantum gravity, viewed as an ordinary quantum field theory. In two dimensions one can solve the lattice theory analytically and the continuum limit is universal, not depending on the details of the lattice regularization.

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1 Introduction

Two-dimensional quantum gravity has been a fruitful laboratory for studying aspects of string theory as well as quantum gravity. One somewhat surprising aspect of Euclidean two-dimension quantum gravity coupled to matter in the form of a conformal field theory, is that the regularized lattice theory, using the so-called dynamical triangulations (DT), can be solved analytically. The details of the DT regularization are unimportant for the continuum limit. In fact it has been a wonderful example of universality in the Wilsonian sense, the critical surface where the continuum limit can be taken being of finite co-dimension in an infinite dimensional coupling constant space (see e.g. [1] for a review). The lattice regularization known as causal dynamical triangulations (CDT) uses a subset of the triangulations used in DT [2, 3]. The original idea was to consider a path integral where spacetime histories before rotating to Euclidean signature were locally causal, i.e. had non-degenerate light cones (see [4] for a review of the CDT approach also in higher dimensions than two). In two dimensions, which is the only case we will consider here, the precise relation between the CDT triangulations and the DT triangulations was described in [5].

There is good evidence of universality of the CDT scaling limit, although one does not have the same comprehensive evidence as for the DT case. First, a related model, in a certain way more general, the so-called string-bit model [6], led to the same scaling limit. Further it was shown in [7] that one could add dimers on the “spatial” CDT links without changing the universality class. Thus it was somewhat surprising that adding further “dressing”, but only along the spatial links, seemingly led to new continuum models, depending on a continuous parameter β (to be defined below) [8]. The purpose of this letter is to show that also for this general set of models one obtains indeed the standard CDT scaling limit.

2 Defining the model

The modified CDT model (not to be mistaken for what has later been called “generalized CDT” [9]) is most easily defined using a lattice dual to the triangulation, i.e. a ϕ^3 graph with a “time” foliation. Fig. 1 shows the dual CDT lattice and its generalization. In this dual picture each vertex represents a triangle in the “original” triangulation and each polygon represents a vertex, the order of which is equal the number of sides in the polygon.

In the modified model one allows a dressing of the horizontal links between two vertical links by rainbow diagrams.

Three coupling constants are assigned to the model: to each vertex one associates a coupling constant g , to a vertex with an incident vertical link an additional

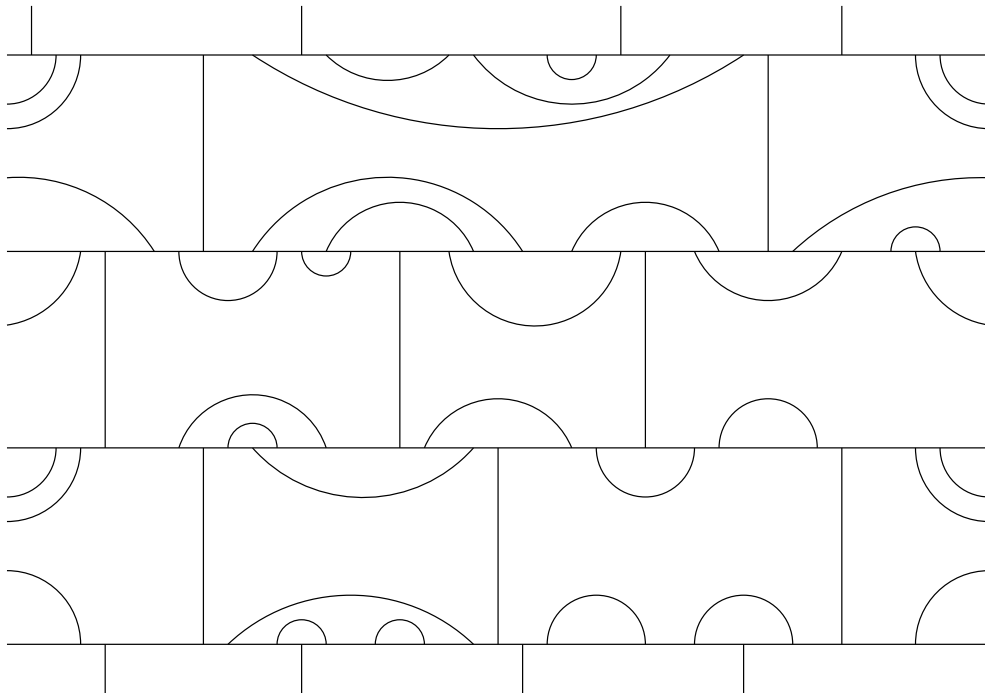


Figure 1: Modified CDT configuration, dual graph.

coupling constant h , and finally to each vertex with an incident rainbow link a coupling constant θ . The parameter

$$\beta = \frac{\theta}{h} \quad (1)$$

governs the density of rainbow links compared to the number of vertical links, i.e. “time-like” links in the original CDT-like ϕ^3 -graph. In this article we will only consider $0 \leq \beta < 1$, which is the range leading to CDT-like theories [8].

As shown in [8] one can define and calculate a transfer matrix for this model. The result is

$$\Theta_{ij} = \sum_k \Theta_{ik}^{(2)} \Theta_{kj}^{(1)} \quad (2)$$

where the index j refers to the number of incoming half-lines which is incident from below on the horizontal line at time t and index k refers to the number of half-lines leaving the horizontal line at time t . Index k plays the same role as index j , only at time-slice $t + 1$. In this way $\Theta_{ik}^{(2)}$ connects outgoing vertical half-lines at t to incoming half-lines at $t + 1$ and $\Theta_{ij}^{(1)}$ incoming half-lines at t to incoming half-lines at $t + 1$.

$\Theta^{(1)}$ is the CDT transfer matrix, already discussed in [2] and analyzed in detail in [7]. If $\theta = 0$ and $h = 1$ there are no rainbow lines and $\Theta^{(2)}$ becomes the identity matrix and Θ also the CDT transfer matrix.

It is convenient to work with the discrete Laplace transforms of Θ , $\Theta^{(1)}$ and $\Theta^{(2)}$. To simplify the expressions somewhat we make the following redefinitions compared to [8]:

$$\Theta_{ij}^{(1)} \rightarrow (2g)^{-i-j} \Theta_{ij}^{(1)}, \quad \Theta_{ij}^{(2)} \rightarrow (2g)^{i+j} \Theta_{ij}^{(2)}. \quad (3)$$

The explicit expressions are then:

$$\Theta^{(1)}(x, y) = \sum_{ij} x^i y^j \Theta_{ij}^{(1)} = \frac{1}{1 - \frac{1}{2}x - \frac{1}{2}y} \quad (4)$$

$$\Theta^{(2)}(x, y) = \frac{C(\hat{x}^2)C(\hat{y}^2)}{(1 - \hat{x}^2 C(\hat{x}^2))(1 - \hat{y}^2 C(\hat{y}^2))(1 - \beta^{-2} \hat{x} \hat{y} C(\hat{x}^2)C(\hat{y}^2))} \quad (5)$$

$$\Theta(x, y) = \oint_C \frac{d\omega}{2\pi i \omega} \Theta^{(1)}(x, \omega^{-1}) \Theta^{(2)}(\omega, y), \quad (6)$$

where the contour encloses cuts and poles and where

$$\hat{x} = 2g\theta x, \quad \hat{y} = 2g\theta y, \quad C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}. \quad (7)$$

Integrating over the simple pole of $\Theta^{(1)}$ one obtains

$$\Theta(x, y) = \frac{1}{1 - \frac{1}{2}x} \frac{C(\bar{x}^2)C(\hat{y}^2)}{(1 - \bar{x}^2 C(\bar{x}^2))(1 - \hat{y}^2 C(\hat{y}^2))} \frac{1}{1 - \beta^{-2} \bar{x} \hat{y} C(\bar{x}^2)C(\hat{y}^2)}, \quad (8)$$

where

$$\bar{x} = \frac{2g\theta}{2 - x} \quad (9)$$

The partition function with open horizontal boundaries after t time steps is¹

$$Z(l, k; t) = \left((\Theta^{(1)}(\Theta^{(2)}\Theta^{(1)})^t)_{kl} \right), \quad (10)$$

and the (discrete) Laplace transformed function is denoted $Z(x, y)$

$$Z(x, y; t) = \sum_{l, k} x^l y^k Z(l, k; t). \quad (11)$$

The partition function after t time steps with periodic boundary conditions in the time direction is

$$Z(t) = \text{tr}(\Theta^t). \quad (12)$$

¹The same continuum limit is obtained by setting $Z(l, k; t) = (\Theta^t)_{kl}$.

3 The continuum limit using the transfer matrix

As shown in [8] the partition function $Z(t)$ has a singularity at

$$\xi_c = 2g\theta \left(\beta + \frac{1}{\beta} \right) = 1. \quad (13)$$

We want to take to continuum limit by approaching this singularity. This is done in the following way [8]:

$$\xi \equiv 2g\theta \left(\beta + \frac{1}{\beta} \right) = 1 - \frac{1}{2}a^2\Lambda \left(\frac{1 - \beta^2}{1 + \beta^2} \right)^2. \quad (14)$$

The interpretation is that a is the lattice spacing, i.e. the link length in the triangulation, and Λ the cosmological constant, such that the average number of triangles is proportional to $1/(\Lambda a^2)$. Thus the average “continuum” area is proportional to $1/\Lambda$.

Until now t has denoted the integer number of time steps in the triangulation. We are interested in a limit where we have a finite continuum time T scaling as

$$T = ta, \quad (15)$$

where a is the lattice spacing defined by (14). We can then write

$$Z(T) = \text{tr } \Theta^t = \text{tr } e^{-TH}, \quad \Theta = e^{-aH}. \quad (16)$$

Thus an expansion of Θ to lowest order in a should allow us to determine H .

If the continuum area is proportional to $1/\Lambda$ we expect the continuum length of a time slice to be proportional to $1/(\Lambda T)$. Thus we expect a scaling $L \propto l a$ where l is the number of space-like links. We can also enforce this on the boundaries:

$$Z(l, k; t) \rightarrow Z(L_0, L_T; T). \quad (17)$$

The discrete Laplace transform of $Z(x, y; t)$ has poles in x, y and it is at these poles one extracts the continuum function $Z(L_0, L_T; T)$. These poles are at $x_c = y_c = 1$ for $a \rightarrow 0$. The terms x^l and y^k in (11) can then be given an interpretation as the part of the action coming from a continuum boundary cosmological term proportional to X if we scale:

$$x = 1 - aX \left(\frac{1 - \beta^2}{1 + \beta^2} \right)^2, \quad L = al \left(\frac{1 - \beta^2}{1 + \beta^2} \right)^2, \quad (18)$$

and thus

$$x^l \rightarrow e^{-LX} \quad \text{for } a \rightarrow 0. \quad (19)$$

With this scaling we obtain a relation similar to (17), going from the discretized expression to the continuum expression:

$$Z(x, y, t) \rightarrow Z(X, Y; T), \quad (20)$$

where the continuum analogue of (11) reads

$$Z(X, Y; T) = \int_0^\infty dL_0 dL_T e^{-L_0 X - L_T Y} Z(L_0, L_T; T). \quad (21)$$

We will return to (17) and (20) in the next section.

We now extract H from $\Theta = e^{-aH}$. It is convenient to use the Laplace transform (6) of Θ . Expanding in a we obtain [8]:

$$\left((1 - aH + O(a^2))\psi \right)(x) = \frac{1}{2} \frac{1 - \beta^2}{1 + \beta^2} \oint \frac{d\omega}{2\pi i \omega} \Theta\left(x, \frac{1}{\omega}\right) \psi(\omega). \quad (22)$$

Here $\psi(\omega)$ is the discrete Laplace transform of a function $\psi(l)$:

$$\psi(\omega) = \sum_l \omega^l \psi(l). \quad (23)$$

The function $\Theta(x, 1/\omega)$ has a pole in ω at 1 for $a \rightarrow 0$ and it has a branch cut located at $\omega \in [-\omega_*, \omega_*]$, where

$$\omega_* = 2 \left(\beta + \frac{1}{\beta} \right)^{-1} + O(a) < 1 \quad \text{for } a \text{ sufficiently small.} \quad (24)$$

We can deform the contour to be a small circle around one and an integration along the branch cut. The integration around $\omega = 1$ allows us to use the expansion (18) for x and ω , and we obtain

$$\oint \frac{dZ}{2\pi i} \left[\frac{1}{Z - X} + \frac{a}{(Z - X)^2} \left(\Lambda + \frac{\beta^2 X^2 - (1 + 3\beta^2) X Z + \beta^2 Z^2}{1 + \beta^2} \right) \right] \psi(Z) + O(a^2), \quad (25)$$

Performing the integration (and ignoring the contribution from the cut) we can identify H as

$$H(X) = (X^2 - \Lambda) \frac{\partial}{\partial X} + X, \quad (26)$$

and by an inverse Laplace transformation

$$H(L) = -L \frac{\partial^2}{\partial L^2} - \frac{\partial}{\partial L} + \Lambda L. \quad (27)$$

This is precisely the ordinary CDT Hamiltonian, the only difference is that in order to obtain it in this form we had to perform a dressing (or renormalization)

of the continuum boundary cosmological constant from a value X , corresponding to $\beta = 0$ to the β dependent value given in (18). This renormalization of X and a similar renormalization of the coupling cosmological coupling constant Λ in (14) is all that is needed to include the effects of the rainbow diagrams.

The contribution from the cut can be written as

$$\tilde{\psi}(x) = \int_{-\omega_*}^{\omega_*} d\omega f(x, \omega) \psi(\omega), \quad (28)$$

where $f(x, \omega)$ is integrable in $[-\omega_*, \omega_*]$ and $\tilde{\psi}(x)$ analytic in the neighborhood of 1 and finite when $a \rightarrow 0$. We cannot view such a function as the Laplace transform of any function $\psi(\sqrt{\Lambda}L)$ depending on the continuum length $L > 0$, the reason being that the inverse Laplace transformation from (26) to (27) gives

$$\int_{i\infty+c}^{i\infty+c} \frac{dX}{2\pi i} e^{XL} \tilde{\psi}(1 - aX) = \delta(L) \tilde{\psi}(1) - a\delta'(L) \tilde{\psi}'(1) + \dots + O(a^n). \quad (29)$$

Thus we do not associate any continuum physics with the analytic function $\tilde{\psi}(x)$ defined by (28) ².

4 The Schwinger representation and the continuum

In [8] the modified CDT Hamiltonian was not derived using the transfer matrix as described above, but rather a so-called Schwinger representation of $Z(x, y; t)$. We now show that this method also leads to (27), i.e. the ordinary CDT Hamiltonian.

The starting point is the following representation of $Z(x, y; t)$ ([8], formula (5.19)):

$$Z(x, y; t) = \prod_{s=0}^t \left(\int_0^\infty d\alpha_s e^{-\alpha_s} \right) e^{\frac{1}{2}(\alpha_0 x + \alpha_t y)} \prod_{r=0}^{t-1} \phi_\beta(g\theta\alpha_r, g\theta\alpha_{r+1}) \quad (30)$$

where

$$\phi_\beta(x, y) = \sum_{k \geq 0} I_k(2x) I_k(2y) / \beta^{2k}. \quad (31)$$

²Of course a function like $\tilde{\psi}(\omega)$ would also not contribute to continuum physics if inserted in (25). The part of a function $\psi(\omega)$ defined as in (23) which *does* contribute to continuum physics in (25) is the part which has a continuum Laplace transform, i.e. the part where $\psi(l)$ in (23) has the form $\psi(\sqrt{\xi - \xi_c}l) \rightarrow \psi(\sqrt{\Lambda}L)$. Since $\sqrt{\xi - \xi_c} \propto a\sqrt{\Lambda}$ it can at most be the tail at infinite l which contributes to continuum physics for a given $\psi(\omega) = \sum_l \omega^l \psi(l)$.

x and y only appears in the exponential function and we can write

$$Z(x, y; t) = \int_0^\infty d\alpha_0 \int_0^\infty d\alpha_t e^{-\frac{1}{2}(1-x)\alpha_0 - \frac{1}{2}(1-y)\alpha_t} F(\alpha_0, \alpha_t; t), \quad (32)$$

where

$$F(\alpha_0, \alpha_t; t) = \left(\prod_{s=1}^{t-1} \int_0^\infty d\alpha_s \right) \prod_{r=0}^{t-1} e^{-(\alpha_r + \alpha_{r+1})/2} \phi_\beta(g\theta\alpha_r, g\theta\alpha_{r+1}). \quad (33)$$

Since $1 - x \propto aX$ and $1 - y \propto aY$, (32) states that in the limit where $a \rightarrow 0$ and thus $Z(x, y; t) \rightarrow Z(X, Y; T)$, $Z(X, Y; T)$ is the Laplace transform of $F(\alpha_0, \alpha_t; t)$, $t = T/a$. Thus, in accordance with (21) we have

$$F(\alpha_0, \alpha_t; t) \propto Z(L_0, L_T; T), \quad (34)$$

where

$$L_0 = \frac{1}{2}a\alpha_0 \left(\frac{1 - \beta^2}{1 + \beta^2} \right)^2, \quad L_T = \frac{1}{2}a\alpha_t \left(\frac{1 - \beta^2}{1 + \beta^2} \right)^2, \quad at = T. \quad (35)$$

If we change variables from α_s to φ_s ,

$$\alpha_s = \frac{\varphi_s^2}{a} \left(\frac{1 + \beta^2}{1 - \beta^2} \right)^2, \quad (36)$$

we obtain

$$Z(L_0, L_T; T) \propto \frac{1}{\sqrt{\varphi_0 \varphi_t}} \int_0^\infty \prod_{s=1}^{t-1} d\varphi_s \prod_{r=0}^{t-1} \frac{\sqrt{\varphi_r \varphi_{r+1}}}{a} \frac{1 + \beta^2}{1 - \beta^2} e^{-\frac{\alpha_r + \alpha_{r+1}}{2}} \phi_\beta(g\theta\alpha_r, g\theta\alpha_{r+1}). \quad (37)$$

The right hand side can be interpreted as a (quantum mechanical) path integral, i.e.

$$\sqrt{\varphi_0 \varphi_t} Z(L_0, L_T; T) \propto \langle \varphi_0 | e^{-TH} | \varphi_t \rangle \quad (38)$$

for some Hamiltonian H . We will now proceed to determine H .

Following [8] we use the notation

$$e^{-\frac{\alpha_0 + \alpha_1}{2}} \phi_\beta(g\theta\alpha_0, g\theta\alpha_1) \sim U_\beta(\alpha_0, \alpha_1) e^{-S_\beta(\alpha_0, \alpha_1)}. \quad (39)$$

According to [8]

$$S_\beta(\alpha_0, \alpha_1) = \frac{1}{2}(\alpha_0 + \alpha_1) - 2g\theta\sqrt{(\alpha_0 + \beta^2\alpha_1)(\alpha_0 + \beta^{-2}\alpha_1)} \quad (40)$$

and

$$U_\beta(\alpha_0, \alpha_1) = \frac{1}{\sqrt{4\pi g\theta}} \frac{1}{((\alpha_0 + \beta^2 \alpha_1)(\alpha_0 + \beta^{-2} \alpha_1))^{1/4}} \times \left(1 + \frac{1}{16g\theta \sqrt{(\alpha_0 + \beta^2 \alpha_1)(\alpha_0 + \beta^{-2} \alpha_1)}} + \dots \right). \quad (41)$$

We now expand in a , with

$$\Delta\varphi = \varphi_1 - \varphi_0 \quad (42)$$

counted as being of order \sqrt{a} as one has to do in a path integral (here we differ from [8]):

$$S_\beta(\alpha_0, \alpha_1) = \frac{\Delta\varphi^2}{2a} - \frac{\beta^2}{(1 + \beta^2)^2} \frac{\Delta\varphi^4}{2a\varphi_0^2} + \frac{a\Lambda}{2} \varphi_0^2 + O(a^{3/2}). \quad (43)$$

We see that we get a standard kinetic term, justifying $\Delta\varphi \propto \sqrt{a}$. (Note that the $\Delta\varphi^4$ term is not present in [8]).

Similarly, we find

$$\frac{\sqrt{\varphi_r \varphi_{r+1}}}{a} \frac{1 + \beta^2}{1 - \beta^2} U_\beta(\alpha_0, \alpha_1) = \frac{1}{\sqrt{2\pi a}} \left(1 + \frac{a}{8\varphi_0^2} - \frac{\beta^2}{(1 + \beta^2)^2} \frac{\Delta\varphi^2}{\varphi_0^2} + O(a^{3/2}) \right). \quad (44)$$

(We note that the $\Delta\varphi^2$ term is not present in [8].)

The Hamilton is finally determined by integrating against a trial state:

$$\begin{aligned} ((1 - aH)\psi)(\varphi_0) &= \int_0^\infty \frac{d\varphi_1}{\sqrt{2\pi a}} e^{-\frac{\Delta\varphi^2}{2a}} \left[1 + \left(\frac{1 - \beta^2}{1 + \beta^2} \right)^2 \frac{a}{8\varphi_0^2} - \frac{\beta^2}{(1 + \beta^2)^2} \frac{\Delta\varphi^2}{\varphi_0^2} \right. \\ &\quad \left. + \frac{\beta^2}{(1 + \beta^2)^2} \frac{\Delta\varphi^4}{2a\varphi_0^2} - \frac{a\Lambda}{2} \varphi_0^2 \right] \left[1 + \Delta\varphi \frac{\partial}{\partial\varphi} + \frac{\Delta\varphi^2}{2} \frac{\partial^2}{\partial\varphi^2} \right] \psi(\varphi_1). \end{aligned} \quad (45)$$

Carrying out the Gaussian integral, we obtain

$$H = -\frac{1}{2} \frac{\partial^2}{\partial\varphi^2} + \frac{\Lambda}{2} \varphi^2 - \frac{1}{8\varphi^2}. \quad (46)$$

This is precisely the CDT Hamiltonian when changing back to the L variable.

5 Critical arches

In principle a new behavior could be possible for $\beta \rightarrow 1$ from below, since in this case the rescaling of lengths and boundary cosmological constants, as defined by

eqs. (18), diverges and it is precisely the limit where the cut will merge with the pole in the expression (8) for Θ . Let us investigate this case by assuming

$$\beta = 1 - a^\eta B, \quad (47)$$

where B is a new physical constant with mass dimension η . To understand the analytic structure of Θ for $a \rightarrow 0$, i.e. $\beta \rightarrow 1$ from below, we expand the argument of the square root related to the Catalan number in the expression for Θ :

$$\sqrt{1 - 4\hat{x}^2} = a^\eta B(1 + aX + \frac{1}{2}a^2(\Lambda - X^2) + O(a^\eta B) + O(a^3)) \quad (48)$$

From this expression it is clear that the cut has disappeared from the expression even though it hits the pole when expressed in terms of unrenormalized variables. To find the Hamiltonian we use the same approach as in Sec. 3, eqs. (22) and (25) and write

$$\tilde{\psi}(x) = (1 - a^\nu H + \dots)\psi(x) := \frac{a^\eta B}{2} \oint \frac{d\omega}{2\pi i \omega} \Theta(x, \frac{1}{\omega})\psi(\omega), \quad (49)$$

where ν is determined by the expansion, We find:

$$\tilde{\psi}(X) = \oint \frac{dZ}{2\pi i} \left[\frac{1 - a^\eta B/2}{Z - X} + a \frac{\Lambda + \frac{1}{2}(X^2 - 4XZ + Z^2)}{(Z - X)^2} \right] \psi(Z). \quad (50)$$

Thus, if $\eta > 1$ we obtain the same results as before (eq. (25) with $\beta = 1$) and if $\eta < 1$ we obtain a trivial Hamiltonian. $\eta = 1$ just adds the positive constant $B/2$ to the CDT Hamiltonian (26). So far we have ignored the contributions from the cut. However, arguments like the ones used in Sec. 3 show that the cut will not contribute in the scaling limit.

6 Discussion

We have shown that the CDT scaling limit is quite universal and independent of details of the lattice regularization, as long as we maintain a reasonable “memory” of the underlying assumed time foliation. Dressing the spatial slices with a few outgrowths should not alter the scaling limit and this is indeed what we have proven to be the case. Potentially there could have been a different behavior in the limit $\beta \rightarrow 1$ where the rainbow diagrams become critical, but explicit calculations showed that it was not the case. The CDT model provides us with a regularized of a theory of fluctuating spacetime which is invariant under spatial diffeomorphisms and which allows for a time foliation. The simplest such continuum model is a Hořava-Lifshitz gravity model in two-dimensions where we only keep terms with at most second order derivatives of the metric, and one can indeed show that such a model has a classical CDT Hamiltonian which when quantized is compatible with the $H(L)$ considered in this paper [10].

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